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The composite operating characteristic under normal and tightened sampling inspection by attributes

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1. Introduction.

Systems of sampling inspection by attributes often contain rules for switching between normal and tightened inspection. The following simple rule proposed by Dodge [1] has been adopted by Military Standard 105 D [3]: (1) When normal inspection is in effect, tightened inspection shall be instituted when 2 out of at most 5 consecutive lots have been rejected. (2) When tightened inspection is in effect, normal inspection shall be instituted when 5 consecutive lots have been accepted. (3) Furthermore the following rule for discontinuation of inspection is specified: In the event that 10 consecutive lots remain on tightened inspection, inspection under the provisions of this document should be discontinued pending action to improve the quality of submitted material.

For single sampling plans Dodge proposes to use the same sample size for normal and tightened inspection and to make the acceptance number one unit smaller for tightened inspection than for normal. In the Military Standard 105 D the reduction is increasing with the acceptance number itself.

The purpose of the present paper is to give a probabilistic description of the effects of these rules. This is done by means of the theory of recurrent events and many of the following results are found by straightforward applications of the methods given by Feller[2]. As far as practical we shall keep to the notation by Feller.

In sections 2 - 6 we shall discuss the effects of using the rules (1) and (2). Modifications resulting from inclusion of rule (3) are treated in sections 7 and 8.

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2. Notation and definitions.

Consider a series of lots submitted for inspection. By inspection of any lot the "inspection system" may be in the state Normal (N) or Tightened (T) and the decision regarding the lot may be Acceptance (a) or Rejection (r). Thus for each lot one of the following four events may occur: N_a, i.e. the lot is accepted under normal inspection, N_r, T_a, T_r. The probabilities of the four events for lot number m are denoted by $p_m^1, \ldots, p_m^4$, where $p_m^1 + \ldots + p_m^4 = 1$.

The probability that the mth lot will be under tightened inspection is $z_m = p_m^3 + p_m^4$.

Corresponding to the sampling plans employed there exist operating characteristics giving the probability of acceptance $P = P(p)$ under normal inspection for lots of quality p (fraction defective in lot), and a corresponding probability P^* under tightened inspection. In the following it is assumed that all lots submitted for inspection are of the same quality. The probabilities P and P^* are considered as known.

From the multiplication rule we have immediately

$$p_m^3 = z_m P^* \quad \text{and} \quad p_m^4 = z_m(1-P^*),$$

and similarly

$$p_m^1 = (1-z_m)P \text{ and } \quad p_m^2 = (1-z_m)(1-P).$$

It follows that the four probabilities are known when z_m is known.

The composite operating characteristic, i.e. the probability of acceptance of a lot of quality p taking into account that some lots are under normal and some under tightened inspection, is

$$p_m^1 + p_m^3 = (1 - z_m)P + z_m P^* \tag{1}$$

which is a weighted average of the two given operating characteristics.

In the following we shall determine z_m and some other important quantities.

Instead of basing the switching rule on (at most) 5 consecutive lots we shall use d lots.

Generating functions of sequences $\{f_m\}$, $\{u_m\}$, etc. will be denoted by F(s), U(s), etc.

3. Results for tightened inspection.

The condition for shifting to normal inspection is that d consecutive lots are accepted under tightened inspection.

Let f_m^N be the probability that the first run of length d of accepted lots occurs at inspection of the mth lot (inspection being tightened of all m lots). Feller [2], p. 300, has shown that the generating function of $\{f_m^N\}$ is

$$F_N(s) = \sum_{m=0}^{\infty} f_m^N s^m = \frac{P^{*d} s^d (1 - P^* s)}{1 - s + Q^* P^{*d} s^{d+1}}, \qquad F_N(1) = 1, \tag{2}$$

$Q^* = 1 - P^*$, and $f_0^N = f_1^N = \ldots = f_{d-1}^N = 0$. Furthermore the mean and variance of the number of lots inspected under tightened inspection are

$$\xi = F_N'(1) = \frac{1 - P^{*d}}{Q^* P^{*d}} \tag{3}$$

and

$$\sigma^2 = \left(\frac{\xi}{1 - P^{*d}} \right)^2 - (2d + 1) \frac{\xi}{1 - P^{*d}} - \frac{P^*}{Q^{*2}} . \tag{4}$$

For $P^* \longrightarrow 1$ we have $\xi \longrightarrow d$ and $\sigma \longrightarrow 0$ whereas for $P^* \longrightarrow 0$ we find $\xi P^* \longrightarrow 1$ and $\sigma/\xi \longrightarrow 1$.

Looking only at lots under tightened inspection the probability u_m^N that tightened inspection ends at the mth lot converges to $1/\xi$ for $m \longrightarrow \infty$.

Table 1 contains some values of ξ and σ as function of P^* for d = 5.

Table 1.

Mean and standard deviation of waiting time (expressed by number of lots inspected) for shift from tightened to normal inspection for d = 5.

P^*	ξ	σ
0.99	5.15	0.76
0.95	5.35	1.94
0.90	6.94	3.24
0.75	12.9	9.31
0.50	62.0	58.2
0.25	136 x 10	136 x 10
0.10	111 x 10^3	111 x 10^3
0.05	337 x 10^4	337 x 10^4
0.01	101 x 10^8	101 x 10^8

4. Results for normal inspection.

The condition for shifting to tightened inspection is that 2 out of at most d consecutive lots are rejected under normal inspection. This is equivalent to the following definition:

The recurrent event E occurs at inspection of the mth lot if and only if one of the d-1 sequences of k letters $N_r N_a \dots N_a N_r$, $k = 2,\dots,d$, occurs with the last N_r at the mth lot and the first N_r not being the last member of a similar sequence resulting in the occurrence of E at the (m-k+1)st lot.
For d = 5 consider as an example the sequence $N_a N_a N_r N_a N_r N_a N_a N_r N_r$ where E occurs at the 5th and the 9th lot but not at the 8th lot since the preceding N_r (at the 5th lot) has already been taken into account in defining the previous occurrence of E.

Let u_m denote the probability that E occurs at inspection of the mth lot.

The probability that E occurs at the (m-k+1)st lot, given that N_r occurs, is u_{m-k+1}/Q since the probability that both E and N_r occur equals the probability that E occurs.

As the events which together define E are mutually exclusive we find the following recursion formulas

$$u_m = \sum_{k=2}^{m} \left(1 - \frac{u_{m-k+1}}{Q}\right) Q^2 P^{k-2}, \quad m = 2,3,\dots,d-1,$$

and

$$u_m = \sum_{k=2}^{d} \left(1 - \frac{u_{m-k+1}}{Q}\right) Q^2 P^{k-2}, \quad m = d,d+1,\dots ,$$

or

$$Q(1-P^{m-1}) = u_m + Q \sum_{k=2}^{m} u_{m-k+1} P^{k-2}, \quad m = 2,3,\dots,d-1, \qquad (5)$$

and

$$Q(1-P^{d-1}) = u_m + Q \sum_{k=2}^{d} u_{m-k+1} P^{k-2}, \quad m = d,d+1,\dots, \qquad (5)$$

which together with $u_o = 1$ and $u_1 = 0$ define $\{u_m\}$. Multiplying (5) by s^m and summing over m we find

$$\sum_{m=2}^{d-1} Q(1-P^{m-1})s^m + Q(1-P^{d-1})s^d/(1-s) = (U(s)-1)(1+Q \sum_{k=2}^{d} P^{k-2}s^{k-1}). \qquad (6)$$

By reduction we get from (6)

$$(1-Ps-QP^{d-1}s^d)/(1-s) = U(s)(1+Qs\,\frac{1-(Ps)^{d-1}}{1-Ps})$$

and

$$F(s) = 1-\frac{1}{U(s)} = Q^2s^2(1-(Ps)^{d-1})/(1-Ps)(1-Ps-QP^{d-1}s^d). \quad (7)$$

From (7) we find $F(1) = 1$, the mean "waiting time"

$$F'(1) = \mu = \frac{1}{Q} + \frac{1}{Q(1-P^{d-1})}\ , \quad (8)$$

and the variance

$$\sigma^2 = F''(1) + \mu - \mu^2$$

$$= \mu^2 + \mu(\frac{2d-2}{1-P^{d-1}} - 2d - 1) - 2\,\frac{1+(d-2)Q}{Q^2(1-P^{d-1})} + \frac{2d}{Q} \quad (9)$$

Looking only at lots under normal inspection the probability u_m^T that normal inspection ends at the mth lot converges to $1/\mu$ for $m \longrightarrow \infty$.

For $P \longrightarrow 1$ we have $\mu \longrightarrow \infty$ and $\sigma/\mu \longrightarrow 1$ whereas for $P \longrightarrow 0$ we find $\mu \longrightarrow 2$ and $\sigma \longrightarrow 0$.

Table 2 contains some values of μ and σ as functions of P for $d = 5$.

Table 2.

Mean and standard deviation of waiting time (expressed by number of lots inspected) for shift from normal to tightened inspection for d = 5.

P	μ	σ
0 99	2638	2635
0.95	128	125
0.90	39.1	36.8
0.75	9.85	7.88
0.50	4.13	2.36
0.25	2 67	0.969
0.10	2.22	0.498
0.05	2.11	0.333
0.01	2.02	0.143

In the following section we shall use the notation $F_T(s), f_m^T$, etc. for the functions defined above in analogy with $F_N(s)$, f_m^N, etc. from the previous section.

5. The probability of tightened inspection and the composite operating characteristic.

In the following we shall assume that normal inspection is used for the first lot.

The probability y_m that normal inspection is reinstated for the first time just after inspection of the mth lot is

$$y_m = \sum_{\nu=1}^{m-1} f_\nu^T f_{m-\nu}^N, \quad y_o = y_1 = y_2 = \ldots = y_{d+1} = 0, \tag{10}$$

so that $Y(s) = F_T(s)F_N(s)$. It is obvious that the waiting time corresponding to the probability distribution $\{y_m\}$ equals the sum of the waiting times for $\{f_m^T\}$ and $\{f_m^N\}$. In particular we have for the average waiting time $Y'(1) = F_T'(1) + F_N'(1) = \mu + \xi$ since $F_T(1) = F_N(1) = 1$.

Let g_m denote the probability that shifting from normal to tightened inspection takes place just after inspection of the mth lot. This requires that tightened inspection is introduced for the first time just after the mth lot or that normal inspection is reinstated for the first time just after the νth lot and shifting to tightened then occurs after inspection of further $m-\nu$ lots, $\nu = 1,2,\ldots,m-1$, i.e.

$$g_m = f_m^T + \sum_{\nu=1}^{m-1} y_\nu g_{m-\nu}, \quad g_o = g_1 = 0, \tag{11}$$

which is the renewal equation. The generating function for $\{g_m\}$ becomes

$$G(s) = F_T(s)/(1-Y(s)) = F_T(s)/(1-F_T(s)F_N(s)) \tag{12}$$

and since $F_T(1) = Y(1) = 1$ it follows from Theorem 1, p. 291, in Feller [2] that

$$g_m \longrightarrow 1/Y'(1) = 1/(\mu + \xi). \tag{13}$$

Denoting by g_m^* the analogously defined probability of shifting from tightened to normal we find

$$g_m^* = y_m + \sum_{\nu=1}^{m-1} y_\nu g_{m-\nu}^*, \quad g_o^* = g_1^* = \ldots = g_{d+1}^* = 0,$$

and $g_m^* \longrightarrow 1/(\mu + \xi)$.

The probability of a shift taking place just after inspection of the mth lot is thus asymptotically equal to $2/(\mu + \xi)$.

We now have for z_m, the probability that the mth lot is under tightened inspection, the first one being under normal inspection,

$$z_m = \sum_{\nu=1}^{m} g_{m-\nu} \sum_{i=\nu}^{\infty} f_i^N , \quad z_o = z_1 = z_2 = 0.$$

Introducing

$$e_m = \sum_{\nu=m}^{\infty} f_\nu^N, \quad m = 0,1,\ldots$$

with the generating function $E(s) = (1-sF_N(s))/(1-s)$ and $E(1) = 1 + \xi$ we find

$$z_m = \sum_{\nu=o}^{m} g_{m-\nu} e_\nu - g_m \tag{14}$$

and

$$Z(s) = G(s)(E(s)-1) = F_T(s)(E(s)-1)/(1-Y(s))$$

$$= sF_T(s)(1-F_N(s))/(1-s)(1-F_T(s)F_N(s)). \tag{15}$$

Since $F_T(1)(E(1)-1)$ is finite and $Y(1)=1$ we find by using the same theorem as above that

$$z_m \longrightarrow F_T(1)(E(1)-1)/Y'(1) = \xi/(\mu + \xi). \tag{16}$$

Table 3 gives lim z_m as function of P and P^* for d = 5.

The composite operating characteristic (1) converges to

$$\overline{P} = (\mu P + \xi P^*)/(\mu + \xi) \tag{17}$$

which is the weighted mean of the operating characteristics for normal and tightened inspection with the average run lengths under normal and tightened inspection as weights.

It follows from the properties of μ and ξ as functions of P and P^* that $\overline{P}$ will be close to P for large P and close to P^* for small P^*, - see Table 3 for details. The switching rule thus produces a composite OC-curve with the desirable property of being steeper than each of the two components.

Table 3.

Limiting probability of tightened inspection for d = 5. ~~and k = 10,~~

P^* \ P	0.99	0.95	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.55	0.50	0.45	0.40	0.30	≤0.20
0.99	0.002														
0.95	0.002	0.044													
0.90	0.003	0.051	0.151												
0.85	0.003	0.061	0.176	0.288											
0.80	0.004	0.074	0.208	0.332	0.432										
0.75	0.005	0.091	0.248	0.384	0.488	0.566									
0.70	0.006	0.114	0.297	0.445	0.551	0.626	0.681								
0.65	0.008	0.146	0.358	0.514	0.618	0.688	0.738	0.775							
0.60	0.011	0.188	0.431	0.590	0.688	0.751	0.793	0.824	0.847						
0.55	0.016	0.247	0.518	0.670	0.757	0.810	0.845	0.869	0.886	0.900					
0.50	0.023	0.327	0.613	0.750	0.822	0.863	0.889	0.907	0.920	0.930	0.938				
0.45	0.035	0.431	0.712	0.824	0.878	0.908	0.926	0.939	0.947	0.954	0.959	0.963			
0.40	0.058	0.558	0.805	0.887	0.923	0.942	0.954	0.962	0.968	0.972	0.975	0.977	0.979		
0.30	0.182	0.821	0.938	0.966	0.978	0.983	0.987	0.989	0.991	0.992	0.993	0.994	0.994	0.995	
0.20	0.597	0.968	0.990	0.995	0.997	0.997	0.998	0.998	0.999	0.999	0.999	0.999	0.999	0.999	0.999
0.10	0.977	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
≤0.01	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

6. An example.

As an example consider the sampling plan for AQL = 2.5 per cent and code letter J in Mil. Std. 105 D. The sample size is 80 and the acceptance number for normal inspection equals 5, for tightened 3. The computations are based on the binomial distribution.

Table 4 gives values of p corresponding to given values of

$$P(p) = \sum_{x=0}^{5} \binom{80}{x} p^x q^{80-x} .$$

For these values of p we compute $P^*(p)$ and from (3) and (8) ξ and μ. Finally the (limiting) composite operating characteristic is found from (17). If we are not interested in the values of ξ and μ we may determine $\xi/(\mu + \xi)$ by interpolation in Table 3.

It will be seen that the upper part of $\bar{P}$ is nearly equal to P and the lower part equal to P^*.

Table 4.

Computation of composite operating characteristic and probability of shift.

P	100p	P^*	μ	ξ	$\bar{P}$	$2/(\mu+\xi)$
0.99	2.23	0.890	2638	7.19	0.990	0.001
0.95	3.32	0.726	128	14.5	0.927	0.014
0.90	3.99	0.604	39.1	28.9	0.774	0.029
0.75	5.30	0.382	9.85	198	0.399	0.010
0.50	7.06	0.175	4.13	729×10	0.176	0.000
0.25	9.14	0.058	2.67	157×10^{4}	0.058	0.000
0.10	11.28	0.016	2.22	969×10^{6}	0.016	0.000
0.05	12.69	0.006	2.11	950×10^{8}	0.006	0.000
0.01	15.57	0.001	2.02	239×10^{13}	0.001	0.000

7. The probability of discontinuation of inspection.

Suppose that the previously discussed rules are supplemented by the following: Inspection is discontinued when k consecutive lots have been under tightened inspection. This is the rule given in [3], and we shall interpret it in the following way: After the kth lot normal inspection shall be reinstated if the last d lots have been accepted, otherwise inspection shall be stopped. It is assumed that k > d.

The result for the mth lot may therefore be N_a, N_r, T_a, T_r or S, where S denotes that "inspection has been discontinued". The corresponding probabilities are denoted by $\hat{p}^1_m$, $\hat{p}^2_m$, $\hat{p}^3_m$, $\hat{p}^4_m$ and $\hat{p}_m$, their sum being 1.

As far as possible we shall use the same notation as previous with the modification that a circumflex denotes that the rule for discontinuation of inspection has been taken into regard.

We therefore have $\hat{z}_m = \hat{p}^3_m + \hat{p}^4_m$, $\hat{p}^3_m = \hat{z}_m P^*$, $\hat{p}^4_m = \hat{z}_m(1-P^*)$, $\hat{p}^1_m = (1-\hat{p}_m-\hat{z}_m)P$, and $\hat{p}^2_m = (1-\hat{p}_m-\hat{z}_m)(1-P)$. It follows that all the probabilities may be found from $\hat{p}_m$ and $\hat{z}_m$, which will be determined in the following.

To handle the stopping rule in a practical way we introduce the truncated distribution

$$\hat{f}^N_m = \begin{cases} f^N_m & \text{for } m = 0,1,\dots,k \\ 0 & \text{for } m = k+1,k+2,\dots, \end{cases} \tag{18}$$

the generating function

$$\hat{F}_N(s) = \sum_{m=1}^{k} f^N_m s^m, \qquad \hat{F}_N(1) = \sum_{r}^{k} f^N_m = \hat{F}_N,$$

and the average waiting time

$$\hat{\xi} = \sum_{m=1}^{k} m f^N_m/\hat{F}_N = \hat{F}'_N(1)/\hat{F}_N. \tag{19}$$

Comparing with (10) it then follows that

$$\hat{y}_m = \sum_{\nu=1}^{m-1} f^T_\nu \hat{f}^N_{m-\nu}, \qquad \hat{y}_o = \hat{y}_1 = \dots = \hat{y}_{d+1} = 0,$$

and

$$\hat{Y}(s) = F_T(s)\hat{F}_N(s), \quad \hat{Y}(1) < 1.$$

Similarly we get analogous to (11)

$$\hat{g}_m = f^T_m + \sum_{\nu=1}^{m-1} \hat{y}_\nu \hat{g}_{m-\nu}, \quad \hat{g}_o = \hat{g}_1 = 0,$$

$$\hat{G}(s) = F_T(s)/(1-F_T(s)\hat{F}_N(s)),$$

and $\hat{G}(1) = 1/(1-\hat{F}_N)$ so that $\hat{g}_m \longrightarrow 0$ for fixed k and $m \longrightarrow \infty$.

Let d_m denote the probability that inspection is discontinued just after inspection of the mth lot. We then find

$$d_m = \hat{g}_{m-k}(1-\hat{F}_N), \quad d_o = d_1 = \dots = d_{k+1} = 0,$$

$D(s) = s^k\hat{G}(s)/\hat{G}(1)$, and $D(1) = 1$. Finally

$$\hat{p}_m = \sum_{\nu=1}^{m-1} d_\nu, \quad \hat{p}_o = \hat{p}_1 = \ldots = \hat{p}_{k+2} = 0, \tag{20}$$

and $\hat{P}(s) = sD(s)/(1-s)$. Since $D(1) = 1$ we have $\hat{p}_m \longrightarrow 1$ for $m \longrightarrow \infty$ and fixed k.

The mean and variance of the number of lots inspected before discontinuation are

$$\eta = (\mu + \hat{\xi}\hat{F}_N + k(1-\hat{F}_N))/(1-\hat{F}_N) \tag{21}$$

and

$$\sigma^2(1-\hat{F}_N) = \sigma_T^2 + \hat{\sigma}_N^2\hat{F}_N + (\mu+\hat{\xi})^2\hat{F}_N/(1-\hat{F}_N) \tag{22}$$

where $\hat{\sigma}_N^2 = \hat{F}_N''(1)/\hat{F}_N + \hat{\xi} - \hat{\xi}^2$.

For $P \longrightarrow 1$ and $P^* \longrightarrow 1$ we have $\eta \longrightarrow \infty$ and $\sigma/\eta \longrightarrow 1$ whereas for $P \longrightarrow 0$ and $P^* \longrightarrow 0$, $\eta \longrightarrow 2+k$ and $\sigma \longrightarrow 0$, see also Table 5.

Table 5.

Mean and standard deviation of waiting time (expressed by number of lots inspected) before discontinuation of inspection for d=5 and k=10.

P	P^*	$\hat{\xi}$	$\hat{F}_N$	η	σ
0.99	0.890	6.07	0.866	19700	19700
0.95	0.726	6.73	0.478	261	249
0.90	0.604	6.99	0.240	63.6	51.5
0.75	0.382	7.27	0.033	20.4	8.65
0.50	0.175	7.42	0.001	14.1	2.38
0.25	0.058	7.48	0.000	12.7	0.970
0.10	0.016	7.49	0.000	12.2	0.498
0.05	0.006	7.50	0.000	12.1	0.333
0.01	0.001	7.50	0.000	12.0	0.143

8. The conditional composite operating characteristic.

Introducing

$$\hat{e}_m = \begin{cases} \sum_{\nu=m}^{\infty} f_\nu^N & \text{for } m = 0,1,\ldots,k \\ 0 & \text{for } m = k+1,k+2,\ldots \end{cases}$$

we get as in (14)

$$\hat{z}_m = \sum_{\nu=0}^{m} \hat{g}_{m-\nu}\hat{e}_\nu - \hat{g}_m, \quad \hat{z}_o = \hat{z}_1 = \hat{z}_2 = 0,$$

and

$$\hat{Z}(s) = \hat{G}(s)\ (\hat{E}(s) - 1),$$

where

$$\hat{E}(s) = \sum_{\nu=0}^{\infty} \hat{e}_\nu s^\nu = (1-s\hat{F}_N(s)-s^{k+1}(1-\hat{F}_N))/(1-s),$$

which lead to

$$\hat{Z}(s) = \frac{s}{1-s}\ \frac{F_T(s)(1-\hat{F}_N(s)-s^k(1-\hat{F}_N))}{1-F_T(s)\hat{F}_N(s)} . \tag{23}$$

For $k \rightarrow \infty$ we have $\hat{Z}(s) \rightarrow Z(s)$ since $\hat{F}_N(s) \rightarrow F_N(s)$.

As $\hat{Z}(1)$ is finite it follows that $\hat{z}_m \rightarrow 0$ in accordance with the previous result that $\hat{p}_m \rightarrow 1$. Similarly we have for the probability of acceptance that $\hat{p}^1_m + \hat{p}^3_m \rightarrow 0$.

It is, however, the conditional probability of acceptance, given that inspection has not been discontinued, which is of interest. <u>The corresponding conditional probability of tightened inspection</u> is $z^*_m = \hat{z}_m/(1-\hat{p}_m)$.

To find the limiting value of z^*_m we need the generating function of $(1-\hat{p}_m)$ which from (20) is found to be

$$\hat{Q}(s) = (1-sD(s))/(1-s)$$

$$= \frac{1-F_T(s)\hat{F}_N(s)-s^{k+1}F_T(s)(1-\hat{F}_N)}{(1-s)(1-F_T(s)\hat{F}_N(s))} . \tag{24}$$

Comparing $\hat{Q}(s)$ and $\hat{Z}(s)$ it will be seen that they may be written as $\hat{Q}(s) = U_1(s)/V(s)$ and $\hat{Z}(s) = U_2(s)/V(s)$ where $U_1(s)$, $U_2(s)$, and $V(s)$ are polynomials, the degree of the U's being lower than the degree of V. According to Feller [2], p. 259, we then have asymptotically

$$s_1^{m+1}\ (1-\hat{p}_m) \sim -U_1(s_1)/V'(s_1)$$

and

$$s_1^{m+1}\ \hat{z}_m \sim -U_2(s_1)/V'(s_1)$$

where s_1 is a root of $V(s) = 0$ which is smaller in absolute value than all other roots.

It follows that $z^*_m \rightarrow U_2(s_1)/U_1(s_1)$ or

$$z^*_m \rightarrow 1-s_1^{-k}(1-\hat{F}_N(s_1))/(1-\hat{F}_N) \tag{25}$$

Table 6.

Limiting conditional probability of tightened inspection for d = 5 and k = 10.

P^* \ p	0.99	0.95	0.90	0.85	0.80	0.75	0.70	0.65	0.60	0.55	0.50	0.45	0.40	0.30	≤0.20
0.99	0.000														
0.95	0.002	0.043													
0.90	0.002	0.048	0.144												
0.85	0.003	0.053	0.158	0.267											
0.80	0.003	0.058	0.172	0.290	0.392										
0.75	0.003	0.062	0.185	0.312	0.422	0.510									
0.70	0.003	0.065	0.196	0.333	0.450	0.543	0.615								
0.65	0.003	0.068	0.206	0.351	0.476	0.575	0.649	0.705							
0.60	0.004	0.071	0.215	0.368	0.500	0.604	0.682	0.739	0.781						
0.55	0.004	0.073	0.222	0.382	0.522	0.632	0.713	0.771	0.813	0.843					
0.50	0.004	0.074	0.227	0.393	0.540	0.655	0.740	0.800	0.841	0.871	0.892				
0.45	0.004	0.07[illegible]	0.231	0.402	0.555	0.675	0.764	0.825	0.867	0.895	0.915	0.929			
0.40	0.004	0.076	0.234	0.409	0.566	0.691	0.783	0.847	0.889	0.917	0.935	0.947	0.956		
0.30	0.004	0.076	0.237	0.416	0.578	0.710	0.808	0.876	0.921	0.948	0.964	0.974	0.980	0.987	
0.20	0.004	0.077	0.238	0.418	0.582	0.717	0.818	0.888	0.935	0.963	0.980	0.988	0.993	0.996	0.998
≤0.10	0.004	0.077	0.238	0.418	0.583	0.718	0.819	0.891	0.938	0.967	0.984	0.992	0.997	0.999	1.000

where s_1 is the smallest root in the equation $1-F_T(s)\hat{F}_N(s) = 0$. By solving this equation numerically one may thus determine $\lim z_m^*$.

Since $\hat{F}_N(s) = P^{*d}s^dR(s)$, where $R(s)$ is a polynomium of degree k-d with coefficients which are nonnegative powers of P^* and Q^* and with the constant term equal to 1, we get from $1-F_T(s)\hat{F}_N(s) = 0$ by means of (7) that

$$Q^2P^{*d}s^{d+2}(1-Ps+(Ps)^2+ \dots + (Ps)^{d-2})R(s)=1-Ps-QP^{d-1}s^d. \qquad (26)$$

Let now $(P,P^*) \longrightarrow (0,0)$. If we suppose that s is bounded, then the left hand side of (26) tends to zero whereas the righthand side tends to one, and therefore we conclude that $s_1 \longrightarrow \infty$. Thus $s_1^{d-k}R(s_1)$ tends to a constant, which means that $s_1^{-k}\hat{F}_N(s_1) \longrightarrow 0$ and hence, from (25), $z_m^* \longrightarrow 1$.

For $(P,P^*) \longrightarrow (1,1)$, suppose that s is bounded. Then the right hand side of (26) tends to 1-s and the left hand side tends to zero. Thus $s_1 \longrightarrow 1$, which means that $(1-\hat{F}_N(s_1))/(1-\hat{F}_N) \longrightarrow 1$ and hence $z_m^* \longrightarrow 0$.

Table 6 shows $\lim z_m^*$ as function of P and P^* for d = 5 and k = 10. Comparing with Table 3 it will be seen that $\lim z_m^* < \lim z_m$ and that there may be considerable differences between the two probabilities. It will be noted that for given P and $P^* \longrightarrow 0$ we have $\lim z_m \longrightarrow 1$ whereas $\lim z_m^*$ tends to a constant less than 1, which result may also be derived from (26).

Table 7 gives a comparison of the limiting composite operating characteristics computed from (16) and (25) for the plan mentioned in section 6. The limiting conditional composite operating characteristic, $\bar{P}^*$, is computed as

$$\bar{P}^* = P^*\lim z_m^* + P(1-\lim z_m^*) .$$

Table 7.

Composite operating characteristics
for d=5 and k=10 computed from (16) and (25).

P	P^*	$\lim z_m$	$\lim z_m^*$	$\bar{P}$	$\bar{P}^*$
0.99	0.890	0.003	0.002	0.990	0.990
0.95	0.726	0.102	0.064	0.927	0.936
0.90	0.604	0.425	0.214	0.774	0.837
0.75	0.382	0.953	0.696	0.399	0.494
0.50	0.175	0.999	0.981	0.176	0.181
0.25	0.058	1.000	1.000	0.058	0.058
0.10	0.016	1.000	1.000	0.016	0.016
0.05	0.006	1.000	1.000	0.006	0.006
0.01	0.001	1.000	1.000	0.001	0.001

To get an idea of the rate of convergence for z_m^* (or any other probability discussed) one may use the recursion formulas. A numerical investigation has shown that z_m^* reaches the limiting values (with 3 decimal places) given in Table 7 before m = 20.

Acknowledgements.

After completion of the present paper our attention has been drawn to a paper by W.R. Pabst [4] which contains a numerical example of a composite operating characteristic derived by P. Martel. The work of Mr. Martel does not seem to have been published. However, Dr. Pabst has kindly put to our disposal the minutes of the meetings of the Working Party for the development of Mil. Std. 105D. Here Mr. Martel has considered the problem corresponding to rules (1) and (2) in section 1 by interpreting the possible "states" of the inspection system, for example the state $(N_r N_a N_a N_a N_r)$, as states of a finite Markov chain with known transition probabilities. The stationary probability distribution of the states is found by solving the corresponding set of linear equations, and the stationary (limiting) probability of acceptance may then be obtained as a linear combination of the solution. No explicit expression has been given but Mr. Martel's numerical results are in agreement with those obtainable from (16).

References.

1. H.F. Dodge: A general procedure for sampling inspection by attributes - based on the AQL concept. ASQC Annual Convention Transactions, 1963, 7 - 19.

2. W. Feller: An introduction to probability theory and its applications I, Wiley, New York, 1957.

3. Military Standard 105 D: Sampling procedures and tables for inspection by attributes. U.S Government Printing Office, Washington, 1963.

4. W.R. Pabst: Mil - Std - 105 D. Industrial Quality Control, 1963, Vol. 20, No. 5, pp. 4 - 9.

Summary.

It is proved that a rule for shifting between normal and tightened inspection as the one defined in Military Standard 105 D leads to a composite operating characteristic which converges to $\overline{P} = (\mu P + \xi P^*)/(\mu + \xi)$, P and P^* being the operating characteristics for normal and tightened inspection respectively, μ and ξ being the average waiting times expressed in number of lots inspected for switching from normal to tightened and from tightened to normal inspection. The average waiting times and the standard deviations are found as functions of P and P^* by means of the theory of recurrent events. Recursion formulas are given for all the probabilities involved.

Introducing furthermore a rule for discontinuation of inspection, analogous formulas are derived and the conditional composite operating characteristic, given that inspection has not been discontinued, is found.

Sommaire.

On a montré qu'une règle pour changer entre l'inspection normale et renforcée, commecelle-ci définie dans Military Standard 105 D, aura pour résultat une courbe d'efficacité composite qui converge vers $\overline{P} = (\mu P + \xi P^*)/(\mu + \xi)$, P et P^* étant les courbes d'efficacité pour l'inspection normale et renforcée respectivement, μ et ξ étant les attentes moyennes pour changer d'inspection normale à renforcée et vice versa. A l'aide de la théorie des événements récurrents, on dérive les attentes moyennes et les écarts-types comme des fonctions de P et P^*. Pour toutes les probabilités dont il s'agit, les formules de récurrence sont déduites.

En outre, en introduisant une règle pour la discontinuation de l'inspection, des formules analogues sont dérivées et la courbe d'efficacité composite conditionelle, supposé que l'inspection n'ait pas été discontinue, est trouvée.